Part1B Advanced Physics

Classical Dynamics

Lecture Handout: 4

Lecturer: J. Ellis

University of Cambridge
Department of Physics

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Please send comments or corrections to Dr.J. Ellis, je102@cus.cam.ac.uk

Contents

4 Rigid Body Dynamics

4.1 Angular Velocity

- Angular velocity is a concept invented so that you can work out how fast any part of a rotating rigid body is moving. The velocity of a part of the body with position vector $\underline{r}$ (measured with respect to some reference point, *O*, *on the axis of rotation*) is given by:

$$\underline{v} = \underline{\dot{R}} + \underline{\omega} \times \underline{r} \qquad 4.1.1$$

where $\underline{\omega}$ is the angular velocity and $\underline{\dot{R}}$ is the velocity of the reference point *O*.

- It is often useful to take the reference point *O* as being the centre of mass, but it need not be. (For example a rolling cylinder, one could usefully take *O* is being on the line of contact between the cylinder and the ground, for which $\underline{\dot{R}}$ is instantaneously zero.)

- Prescription for finding $\underline{\omega}$ for a rigid body.

(1) decide on an inertial frame of reference in which you wish to view the body – this corresponds to setting up what you are going to take as $\underline{\dot{R}}$. (For example with a rolling cylinder you could take the ground's reference frame in which case $\underline{\dot{R}}$ is the velocity of the centre of mass as it rolls along, or you could take one in which the centre of mass was stationary in which case $\underline{\dot{R}}$ is zero.)

(2) take two photos of the body a very short time apart.

(3) Look for the points on the body that do not move – these will lie on the axis of rotation and will define the direction of $\underline{\omega}$. For the rolling cylinder, if you take your reference frame as that of the ground – then the axis of rotation will be the line of contact between the cylinder and the ground, since this line on the cylinder is instantaneously stationary. If you take the centre of mass frame, then the axis will be the central line of the cylinder.

(4) Choose a point on the axis of rotation to be your reference point.

(5) Find the magnitude of $\underline{\omega}$ such that the velocities of each point on the cylinder (as determined by how far they have moved between the two photos) is given by:

$$\underline{v} = \underline{\dot{R}} + \underline{\omega} \times \underline{r}$$

4.2 Addition of angular velocities

- Suppose you have a top spinning about its axis with angular velocity $\underline{\omega}$, resting on one end with its axis at some angle to the vertical ($\underline{\omega}$ is directed along the axis of the top). Under the action of gravity it will precess about the vertical axis with an angular velocity $\underline{\Omega}$. We will take as the reference point O, the point of contact with the ground – since this point is on both axes of rotation. If one looks at the body in a frame rotating around with it as it precesses, then all you can see is the spinning of the top and the velocity of a point on the top is given by $\underline{\omega}\times\underline{r}$. If one now steps out of the rotating frame into the non –rotating one, in which you can see the precession, we need to add in the velocity of the point on the body due to the rotation of the whole top, which is given by $\underline{\Omega}\times\underline{r}$. The total velocity of the point is therefore give by $(\underline{\omega}+\underline{\Omega})\times\underline{r}$ (note this only works because the origin of the $\underline{r}$ vector is taken as the point of intersection of the $\underline{\omega}$ and $\underline{\Omega}$ vectors). If you want an angular velocity to use in the prescription for the velocity of any point of the body (equation 4.1.1), then you would simply use $\underline{\omega}+\underline{\Omega}$, i.e. the angular velocities have added as vectors. It is hard to visualise the addition of angular velocities, and even harder to visualise the decomposition of an angular velocity into components – but if one takes the (correct) view that angular velocity is just something invented to give a prescription for velocity, and that as such is behaves as a vector then one can still use it very successfully.

- In general if a rigid body is rotated by an angle α about one axis and then an angle β about a second the body will end up in a different orientation to that obtained by first rotating the body an angle β about the second axis and then an angle α about the first. However, in the limit of small angles, we can show that the order in which the rotations are made does not matter and the end result can be written as linear combination of the two.

Consider a rotation by an angle α about the x axis. In Cartesian coordinates the rotation may be represented by a matrix:

$$\underline{\underline{A}}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}$$

and a rotation β about the y axis may be represented by a matrix:

$$\underline{\underline{B}}=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}$$

so if the A rotation is first performed and then the B rotation, the effect may be represented by the matrix:

$$\underline{\underline{B}}\,\underline{\underline{A}}=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}=\begin{pmatrix}\cos\beta & \sin\beta\sin\alpha & \sin\beta\cos\alpha\\ 0 & \cos\alpha & -\sin\alpha\\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha\end{pmatrix}$$

and when performed in the reverse order, the rotations may be represented by:

$$\underline{\underline{A}}\,\underline{\underline{B}}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta\\ -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta\end{pmatrix}$$

Clearly $\underline{\underline{A}}\,\underline{\underline{B}}\neq\underline{\underline{B}}\,\underline{\underline{A}}$, but if we go to the limit of small angles, for which the first order limiting approximations $\cos\alpha=1$ and $\sin\alpha=\alpha$ may be made, then we see that:

$$\underline{\underline{A}}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -\alpha\\ 0 & \alpha & 1\end{pmatrix},\qquad \underline{\underline{B}}=\begin{pmatrix}1 & 0 & \beta\\ 0 & 1 & 0\\ -\beta & 0 & 1\end{pmatrix}$$

and, keeping only terms that are first, or zeroth order we see that the order in which the infinitesimal rotations are performed does not affect the result:

$$\underline{\underline{A}}\,\underline{\underline{B}}=\underline{\underline{B}}\,\underline{\underline{A}}=\begin{pmatrix}1 & 0 & \beta\\ 0 & 1 & -\alpha\\ -\beta & \alpha & 1\end{pmatrix}$$

The change in position vector produced by the two rotations is then:

$$\delta\underline{r}=\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{r}-\underline{r}=\left(\underline{\underline{A}}\,\underline{\underline{B}}-\underline{\underline{I}}\right)\underline{r}=\begin{pmatrix}0 & 0 & \beta\\ 0 & 0 & -\alpha\\ -\beta & \alpha & 0\end{pmatrix}\underline{r}$$

and if a small additional rotation of γ about the z axis is made, and the infinitesimal rotations written as $\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}=\underline{\omega}\,\delta t$ then:

$$\delta\underline{r}=\begin{pmatrix}0 & -\gamma & \beta\\ \gamma & 0 & -\alpha\\ -\beta & \alpha & 0\end{pmatrix}\underline{r}=\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\times\underline{r}=\begin{pmatrix}\delta t\omega_x\\ \delta t\omega_y\\ \delta t\omega_z\end{pmatrix}\times\underline{r}=\delta t\underline{\omega}\times\underline{r}$$

and

$$\frac{\delta \underline{r}}{\delta t} = \underline{v} = \underline{\omega} \times \underline{r}$$

Thus:

- we prove the formula that relates the velocity and angular velocity
- we prove that angular velocity ands as a vector and can be decomposed into a sum of components as a vector.
- we show that addition, or decomposition of angular velocity may be visualised by considering small actual rotations.

4.3 Tensor of inertia

- Angular momentum of a rigid body:

$$\begin{aligned} \underline{J} &= \sum \underline{r} \times \underline{p} = \sum \underline{r} \times m(\underline{\omega} \times \underline{r}) \\ &= \sum m r^2 \underline{\omega} - \sum m \underline{r}(\underline{\omega} . \underline{r}) \end{aligned}$$

- Consider the x component of $\underline{J}$:

$$\begin{aligned} J_x &= \sum m r^2 \omega_x - \sum m(\omega_x x + \omega_y y + \omega_z z) x \\ &= \sum m(y^2 + z^2)\omega_x - \sum mxy\omega_y - \sum mxz\omega_z \end{aligned}$$

- so for $\underline{J}$:

$$\underline{J} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\underline{J} = \underline{\underline{I}}\ \underline{\omega}$$

where $\underline{\underline{I}}$ is the inertia tensor, composed of the moments of inertia (on the leading diagonal) and products of inertia (off diagonal elements). Note that $\underline{\underline{I}}$ is a symmetrical tensor – which has important consequences outlined below.

4.4 Principal axes

- In general $\underline{J}$ is not parallel to $\underline{\omega}$, but we can look for special cases where it is:

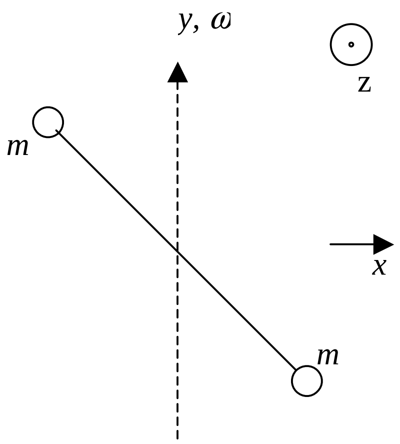
$$\underline{J} = \underline{\underline{I}}\ \underline{\omega} = \lambda \underline{\omega}$$

i.e. we are looking for eigenvectors of the inertia tensor, and the corresponding eigenvalues λ, which will give the moments of inertia about the eigenvectors.

- since $\underline{\underline{I}}$ is a symmetric tensor in 3-D, then it will have 3 eigenvectors, which will be mutually orthogonal (or can be made to be mutually orthogonal in the case of eigenvectors whose corresponding eigenvalues are degenerate). i.e. for ANY rigid body will one can always find three mutually perpendicular axes for which $\underline{J}$ is parallel to $\underline{\omega}$ if $\underline{\omega}$ is aligned with one of the axes. It therefore usually makes sense to choose these three *principal axes* (labelled 1,2,3 by convention) as the reference directions for the coordinate system since in this coordinate system the inertia tensor will be diagonal.

$$\underline{J} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

- for example – rotating dumbbell, length 2*a*, with its axis at 45º to the axis of rotation



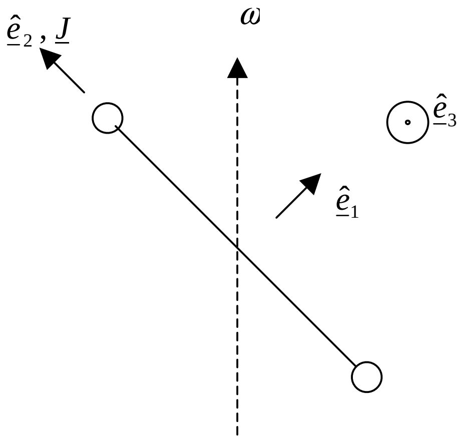
Using x, y, *z*, coordinates we have:

masses at $\begin{pmatrix} a/\sqrt{2} \\ -a/\sqrt{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -a/\sqrt{2} \\ a/\sqrt{2} \\ 0 \end{pmatrix}$

giving:

$$\underline{\underline{I}} = \begin{pmatrix} ma^2 & ma^2 & 0 \\ ma^2 & ma^2 & 0 \\ 0 & 0 & 2ma^2 \end{pmatrix}$$

The eigenvectors of $\underline{\underline{I}}$ are along the directions: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ about which the moments of inertia are: $I_1 = 2ma^2$, $I_2 = 0$, $I_3 = 2ma^2$



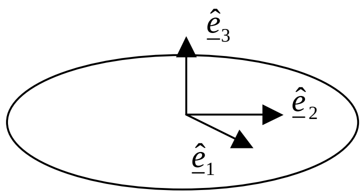
With respect to the principal axes, the angular momentum is then given by:

$$\underline{J} = \begin{pmatrix} 2ma^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2ma^2 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 2ma^2\omega_1 \\ 0 \\ 2ma^2\omega_3 \end{pmatrix}$$

so, for $\underline{\omega}$ is indicated: $\underline{J} = \begin{pmatrix} \sqrt{2}ma^2\omega \\ 0 \\ 0 \end{pmatrix}$

Other examples of the inertia tensor:

Disk:

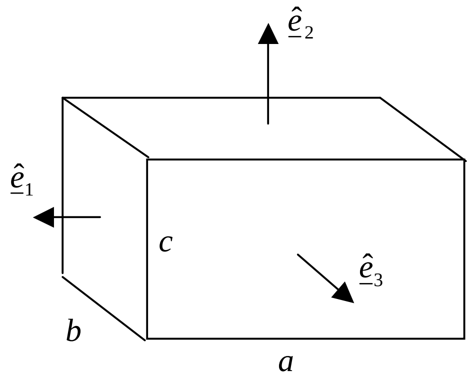


Radius *a*, Mass *m* $\quad \underline{\underline{I}} = \frac{1}{4}ma^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(By convention the axis of symmetry is index 3)

Block Mass *m*

$$\underline{\underline{I}} = \frac{1}{12}m\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+b^2 & 0 \\ 0 & 0 & a^2+c^2 \end{pmatrix}$$



4.5 Rotational Kinetic Energy

- Rotational kinetic energy given by:

$$T = \frac{1}{2}\sum m(\underline{\omega}\times\underline{r}).(\underline{\omega}\times\underline{r}) = \frac{1}{2}\sum m\underline{\omega}.(\underline{r}\times(\underline{\omega}\times\underline{r})) = \frac{1}{2}\underline{\omega}.\sum(\underline{r}\times\underline{p}) = \frac{1}{2}\underline{\omega}.\underline{J} = \frac{1}{2}\underline{\omega}.\underline{\underline{I}}\underline{\omega}$$

- with respect to the principal axes:

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \qquad 4.5.1$$

- consider keeping K.E. constant, but varying $\underline{\omega}$. Equation 4.5.1 maps out a surface of constant K.E in $\underline{\omega}$ space – i.e. the tip of the $\underline{\omega}$ vector maps out an ellipsoid in space – the inertia ellipsoid. Equation 4.5.1 may be rewritten as:

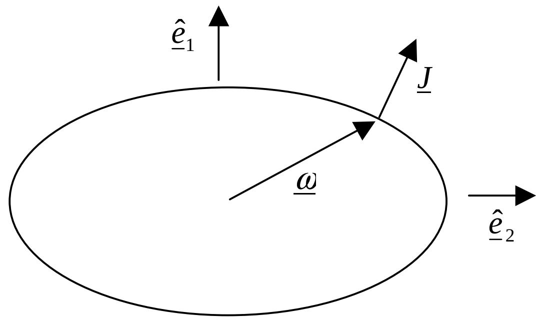
$$\frac{\omega_1^2}{a_1^2} + \frac{\omega_2^2}{a_2^2} + \frac{\omega_3^2}{a_3^2} = 1 \qquad \text{where } a_1 = \sqrt{\frac{2T}{I_1}} \qquad etc$$

- The angular momentum vector associated with a particular $\underline{\omega}$ is perpendicular to the surface of the ellipsoid at the point where the $\underline{\omega}$ vector touches the ellipsoid. To prove we use the fact that for any $\underline{\omega}$ in the surface of the ellipsoid, the T remains fixed, i.e.

$$\delta T = 0 = I_1\omega_1\delta\omega_1 + I_2\omega_2\delta\omega_2 + I_3\omega_3\delta\omega_3 = \underline{J}.\delta\underline{\omega}$$

and $\underline{J}$ is perpendicular to any vector in the surface of the ellipsoid.

Drawing a section through the ellipsoid:



4.6 Euler's Equations

- Principal axes are fixed to the Body – define a *Body Frame of Reference*.

- Axes fixed in the laboratory define a *Space Frame of Reference*.

- c.f. section 1.4.1 'Transformations from stationary to rotating frames' and formula for working out a true time derivative in terms of the coordinates for a vector in a rotating frame.

$$\left[\frac{d\underline{A}}{dt}\right]_{S_0} = \left[\frac{d\underline{A}}{dt}\right]_S + \underline{\omega}\times\underline{A} \qquad 1.4.1$$

- The principal axes are often the most convenient coordinate system for describing rotational motion – but they are themselves rotating at $\underline{\omega}$ with respect to the space frame, and we need equation 1.4.1 to give true time derivatives.

- The dynamics of a rotating system are determined by the relation:

$$\dot{\underline{J}} = \underline{G}$$

where $\underline{J}$ is the total angular momentum of the system and $\underline{G}$ is the total external couple on the system and so we need to use 0.1 to give us $\underline{\dot{J}} = \underline{G}$ in terms of coordinates in the body frame:

$$\underline{G} = \left[\frac{d\underline{J}}{dt}\right]_{Space} = \left[\frac{d\underline{J}}{dt}\right]_{Body} + \underline{\omega}\times\underline{J} \qquad 4.6.1$$

$$(\underline{\omega}\times\underline{J})_1 = \omega_2 J_3 - \omega_3 J_2 = \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2 = \omega_2\omega_3(I_3 - I_2) \qquad etc$$

$$\boxed{\begin{aligned} G_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ G_2 &= I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\ G_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{aligned}}$$

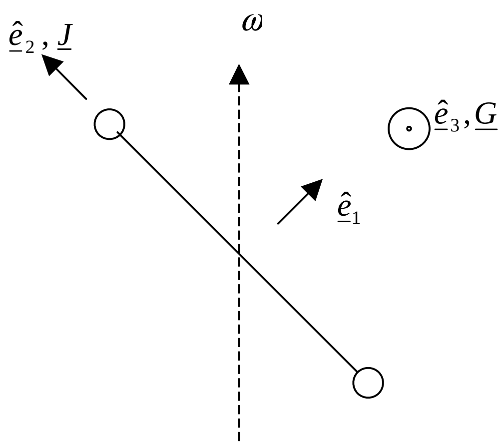
Euler's Equations

- e.g. *Consider dumbbell*

$\underline{\omega} = \left(\omega/\sqrt{2} \quad \omega/\sqrt{2} \quad 0\right)$ fixed in body $\Rightarrow \underline{\dot{\omega}} = \underline{0}$

$I_1 = 2ma^2,\ I_2 = 0,\ I_3 = 2ma^2$

Euler's Equations gives: $\underline{G} = \begin{pmatrix} 0 \\ 0 \\ ma^2\omega^2 \end{pmatrix}$



4.7 Free Precession

- When there is no applied torque, $\underline{G} = \underline{0}$, the motion is described as free precession. (e.g. rotation of a book/asteroid in free fall/orbit around Sun)

- For simplicity take case of a symmetrical top, i.e. $I_1 = I_2 \neq I_3$

- Euler's equations become:

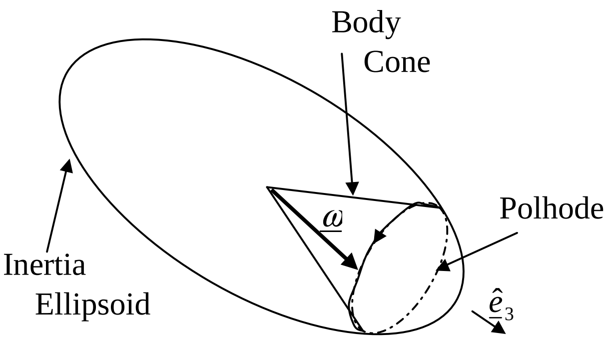
$$\begin{aligned} I_1\dot{\omega}_1 &= +\omega_2\omega_3(I_1 - I_3) \\ I_2\dot{\omega}_2 &= -\omega_3\omega_1(I_1 - I_3) \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

$\Rightarrow$ ω_3 remains constant and defining $\frac{I_1 - I_3}{I_1}\omega_3 = \Omega_b$, the *body frequency* gives:

$$\dot{\omega}_1 = \Omega_b\omega_2 \quad \text{and} \quad \dot{\omega}_2 = -\Omega_b\omega_1 \quad \Rightarrow \quad \ddot{\omega}_1 = -\Omega_b^2\omega_1 \quad \text{and} \quad \ddot{\omega}_2 = -\Omega_b^2\omega_2$$

with a general solution:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A\begin{pmatrix} \sin(\Omega_b t + \phi) \\ \cos(\Omega_b t + \phi) \end{pmatrix}$$



-Thus in the body frame $\underline{\omega}$ precesses around the 3 axis with angular velocity $\underline{\Omega}_b$

- Surface traced out by the angular velocity vector is known as the *body cone*.

- The curve traced out on the inertia ellipsoid by the angular velocity vector is known as the *Polhode*.

The body axes move with respect to the space frame such that:

(a) Instantaneously the whole body rotates about $\underline{\omega}$. (an instant later, $\underline{\omega}$ has moved on around the polhode and we have a new axis of rotation).

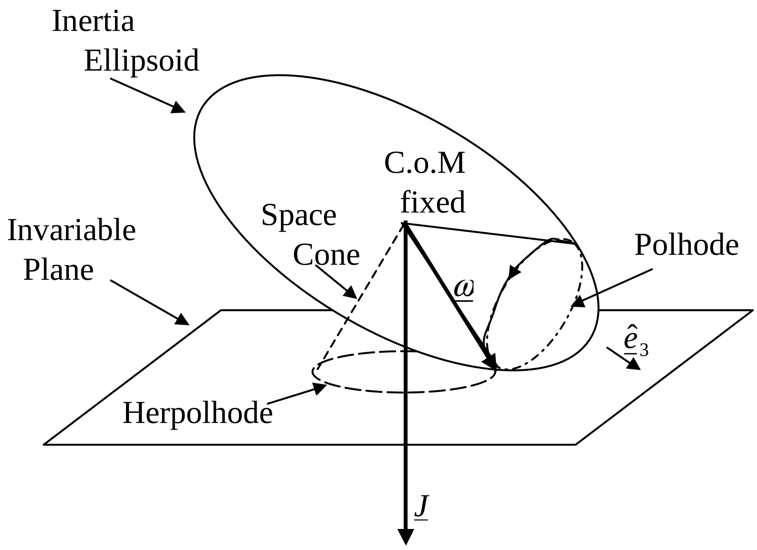
(b) Instantaneously the line on the body that coincides with $\underline{\omega}$ is stationary.

(c) The body moves such that $\underline{J}$, which is given by the perpendicular to the inertia ellipsoid at the point where $\underline{\omega}$ touches it, remains of constant direction and size in space ($\dot{\underline{J}} = \underline{G} = \underline{0}$)

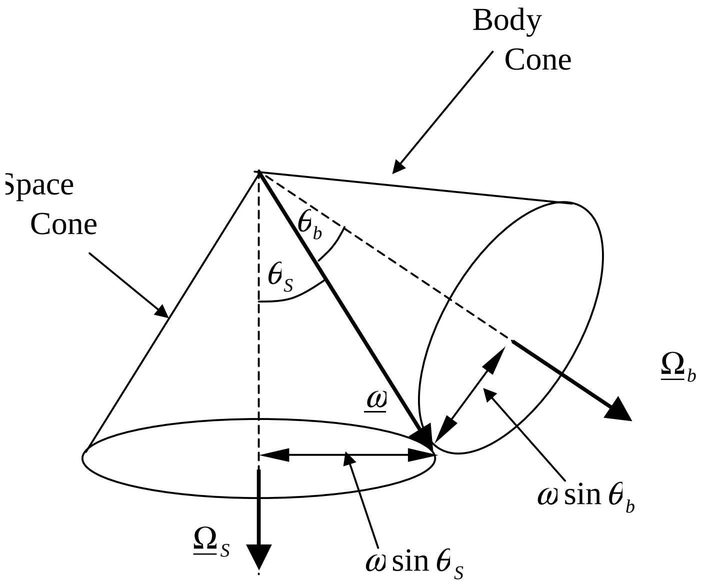
(d) Since $T = \frac{1}{2}\underline{\omega}\cdot\underline{J}$ is constant, and ω and $\underline{J}$ are also constant, the angle between $\underline{\omega}$ and $\underline{J}$ is also constant and $\underline{\omega}$ precesses around $\underline{J}$.

- This motion can be represented by Poinsot's Construction.

4.8 Poinsot's Construction (non examinable)



- Draw a plane (the invariable plane) tangential to the inertia ellipsoid a the point where $\underline{\omega}$ touches the ellipsoid. $\underline{J}$ is perpendicular this plane. Since $\underline{\omega}$ is instantaneously at rest and since the inertial ellipsoid rotates around $\underline{\omega}$ and since the angle between $\underline{\omega}$ and $\underline{J}$ is constant, the inertia ellipsoid rolls on the invariable plane with $\underline{\omega}$ tracing out *space cone* is ellipsoid rolls.



- The body cone is rolling around the space cone

$$\Rightarrow \qquad \Omega_S \omega \sin\theta_S = \Omega_b \omega \sin\theta_b$$
$$\Rightarrow \qquad \underline{\omega} \times \underline{\Omega}_S = -\underline{\omega} \times \underline{\Omega}_b$$

where $\underline{\omega} = (\omega_1 \quad \omega_2 \quad \omega_3)$

$$\underline{\Omega}_B = (0, \quad 0 \quad \Omega'_B)$$
$$\underline{\Omega}_S = \frac{\Omega_S}{J}\underline{J} = \frac{\Omega_S}{J}(I_1\omega_1, \quad I_2\omega_2, \quad I_3\omega_3)$$

hence:

$$\underline{\omega} \times \underline{\Omega}_S = \frac{\Omega_S}{J}(\omega_2\omega_3(I_3 - I_1), \quad \omega_1\omega_3(I_1 - I_3), \quad 0)$$
$$\underline{\omega} \times \underline{\Omega}_b = \Omega_b(\omega_2, \quad -\omega_1, \quad 0)$$
$$\Rightarrow \qquad \Omega_S = \Omega_b J/[\omega_3(I_1 - I_3)] = J/I_1$$

- $\underline{\omega}$ and $\underline{\hat{e}}_3$ precess at same space frequency $\Omega_S = J/I_1$ around direction of $\underline{J}$.

Example: Polar/Chandler wobble of Earth: $(I_3 - I_1)/I_1 = \beta \approx 1/300$ Oblate; θ_S negative. Angle tiny - $\theta_S \approx -\beta\theta_b \approx -\theta_b/300$. Space cone tiny – inside body cone which swings around it each day. In about 300 days, ω should move in a cone round $\underline{\hat{e}}_3$. $\underline{J}$-$\underline{\hat{e}}_3$. Period is actually around 427 days and irregular because the Earth is not rigid. Its amplitude varies around 0.2 seconds, 3 -15m. The wobble is significant for satellite navigations systems, and may be linked to tectonic activity.

4.9 Stability of precession and the asymmetric top

- In general $I_1 \neq I_2 \neq I_3$ - asymmetric top. Consider $\underline{\omega}$ close to $\hat{\underline{e}}_3$ axis, so that $\omega_1,\ \omega_2 << \omega_3$.

Euler's equations give:

$$I_3\dot{\omega}_3 = -\omega_1\omega_2\left(I_2 - I_1\right)$$

and $\dot{\omega}_1,\ \dot{\omega}_2 >> \dot{\omega}_3$. Thus ω_3 is approximately constant

$$I_1\dot{\omega}_1 = -\omega_2\left[\omega_3\left(I_3 - I_2\right)\right]$$
$$I_2\dot{\omega}_2 = +\omega_1\left[\omega_3\left(I_3 - I_1\right)\right]$$

with expressions in square brackets approximately constant.

Now try solution $\omega_1 = A\cos\Omega_b t$, $\omega_2 = B\sin\Omega_b t$, so:

$$I_1 A\Omega_b = B\omega_3\left(I_3 - I_2\right)$$
$$I_2 B\Omega_b = A\omega_3\left(I_3 - I_1\right)$$

and:
$$\Omega_b^2 = \omega_3^2\,\frac{\left(I_3 - I_1\right)\left(I_3 - I_2\right)}{I_1 I_2}$$

Thus we have stable precession (Ω_b real) when $I_3 > I_{1,2}$ or $I_3 < I_{1,2}$ - i.e. when axis of precession corresponds to greatest or least principal moment of inertia. Otherwise Ω_b imaginary and have unstable precession.

- If the orientation of a satellite is important, then it must not be set rotating about the intermediate axis of inertia because apparently chaotic motion results that is very hard to control. (The motion is actually well defined, but complex and is not actually chaotic.)

- What path does $\underline{J}$ trace out w.r.t. principal axes?- consider conservation of angular momentum:

$$J^2 = J_1^2 + J_2^2 + J_3^2 = \text{constant} \qquad \text{(sphere in } \underline{J} \text{ space)}$$

- consider conservation of energy:

$$T = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3} = \text{constant} \qquad \text{(the 'Binet' ellipsoid)} \qquad 4.9.1$$

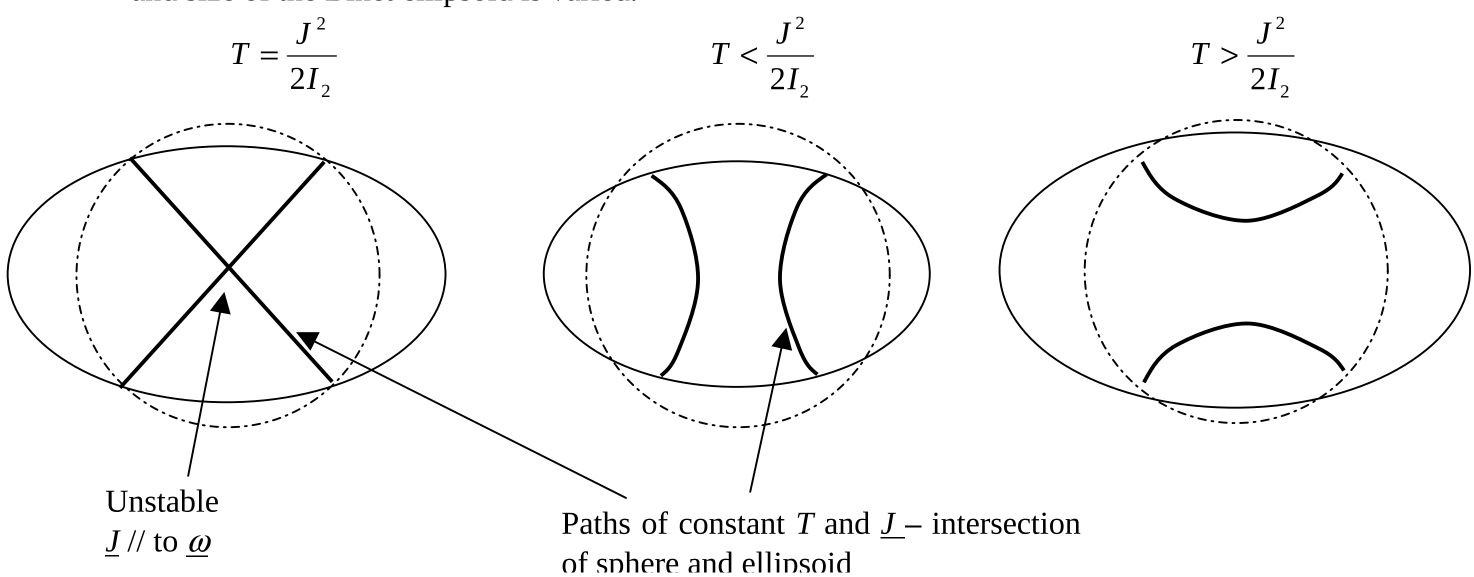
$\Rightarrow$ $\underline{J}$ must follow a path that lies on the intersection of these two surfaces.

4.10 Details of the free precession of an asymmetric top (non-examinable)

- The rate at which $\underline{J}$ moves along the line of intersection of the J^2 sphere and the Binet ellipsoid is given by Poinsot's construction – *i.e.* by a consideration of how $\underline{\omega}$ behaves. (Remember that Poinsot's construction uses the inertial ellipsoid, i.e. T given in terms of $\underline{\omega}$)

- For a symmetrical top the line of intersection of the J^2 sphere and the Binet ellipsoid is a circle with its plane perpendicular to the 3 axis.

- If $\underline{J}$ is perfectly aligned along a principal axes, then its direction does not change with time. However, if it is slightly misaligned, then we showed in section 4.9 that when $\underline{J}$ is close to a principal axis either of greatest or of least principal moment of inertia, $\underline{J}$ performs stable precessions, moving around an ellipse with respect to the body coordinates, but if $\underline{J}$ is close to the principal axis of intermediate moment of inertia, unstable precession results, with large and sometimes rapid changes of direction of $\underline{J}$.
- If $\underline{J}$ is exactly aligned with the principal axis with the largest moment of inertia, the J^2 sphere lies entirely outside the Binet ellipsoid, except at the $\underline{J}$ axis, where the surfaces touch. Increasing the energy slightly makes the path of intersection an ellipse around the principal axis – giving stable precession.
- Similarly - if $\underline{J}$ is aligned with the principal axis with the smallest moment of inertia, the J^2 sphere lies entirely inside the Binet ellipsoid, except at the $\underline{J}$ axis, where the surfaces touch. Decreasing the energy slightly makes the path of intersection an ellipse around the principal axis – again giving stable precession.
- The figure below is drawn looking down the intermediate axis and deals with the case in which $\underline{J}$ is close to the intermediate axis . The solid line is the Binet ellipsoid and the dashed line the J^2 sphere. If $\underline{J}$ is perfectly aligned with the intermediate axis, then as one goes towards the axis of least moment of inertia, the J^2 sphere has a radius of curvature greater that that of the Binet ellipsoid so the sphere lies outside the ellipsoid, and as one goes towards the axis with greatest moment of inertia the J^2 sphere has a radius of curvature smaller than that of the Binet ellipsoid. This geometry results in complex paths of intersection as the kinetic energy and size of the Binet ellipsoid is varied.



4.11 The major axis theorem for non rigid bodies.

- So far we have assumed that the rotating body is perfectly rigid, and as such does not bend or deform whilst the axis of rotation moves around the body in free precession. However, as the axis of rotation moves around the body, the centrifugal forces acting on different parts of the body change and its elastic deformation will change with time.

- If the body is not perfectly elastic, then as the deformation changes, macroscopic potential and kinetic energy will be converted to heat, and the macroscopic kinetic energy of the body decreases. Therefore, unless a body is rotating with $\underline{\omega}$ perfectly aligned with a principal axis (which gives stable rotations with $\underline{\omega}$ unchanging with time), the body is continuously loosing kinetic energy, and the Binet ellipsoid (equation 4.9.1) slowly shrinks with time.

- As previously, in the frame of reference of the body, *i.e.* in the frame of reference of the principal axes, $\underline{J}$ must follow a path that lies on the intersection of the J^2 sphere and the Binet ellipsoid.

- Suppose we start with $\underline{J}$ close to the principal axis with the smallest moment of inertia (the minor axis). The J^2 sphere lies mainly inside the Binet ellipsoid, with the two surfaces cutting on an ellipse that lies around the minor axis, and $\underline{J}$ moves around this ellipse. As time progresses, kinetic energy is lost, the Binet ellipsoid shrinks and the ellipse describing the motion of $\underline{J}$ increases in size and moves out from the minor axis. If the body is asymmetric then there comes a point when $\underline{J}$ approaches the intermediate axis and its precession becomes unstable and apparently chaotic. As further energy is lost, and the Binet ellipsoid shrinks further, $\underline{J}$ starts to move towards the principal axis with the largest moment of inertia (the so called major axis), returning to stable elliptical precessions of ever decreasing size as it approach the major axis. When $\underline{J}$ finally arrives at the major axis, then the Binet ellipsoid can shrink no further because is now entirely inside the J^2 sphere and conservation of momentum dictates that $\underline{J}$ must lie on the J^2 sphere. $\underline{J}$ can only move away from the major axis if additional kinetic energy is given to the body, and so the body settles in a state of uniform and constant rotation about the major axis. As such the centrifugal forces on the body do not change with time, and so there are no further changes in the shape of the body with time and so no mechanism for further energy loss – which helps us see why in practice no further energy is lost once $\underline{J}$ has aligned itself with the major axis.

- We therefore have the 'major axis theorem' for freely rotating bodies:

’Any freely rotating body that is not perfectly rigid, will loose kinetic energy and whist its angular momentum remains constant in space, it moves with respect to the body until the body is rotating about its major axis.’

4.12 Examples of the major axis theorem

Celestial Objects

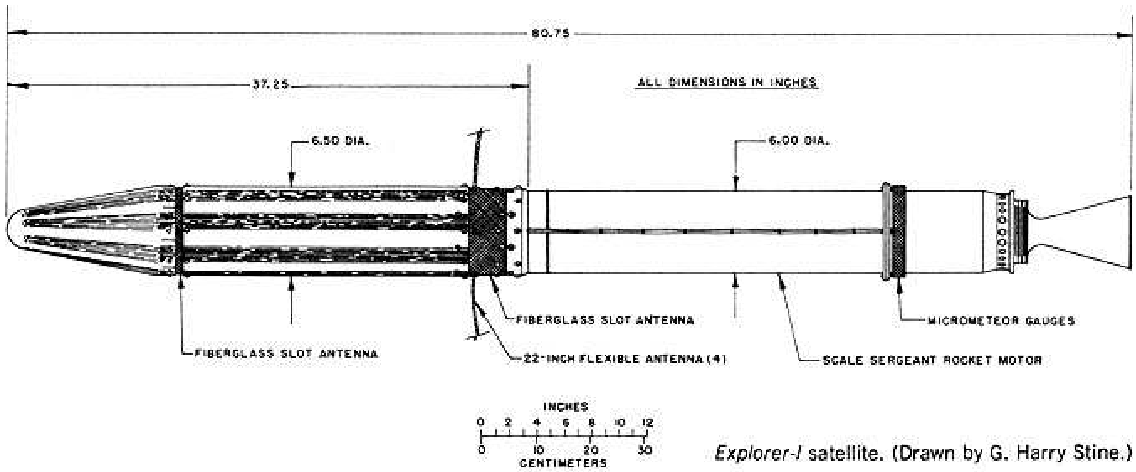
All known freely rotating celestial objects rotate about an axis their major axes – be they asteroids, galaxies or planets like the Earth.

Chandler Wobble

The Earth is far from rigid, and so any free precession will decay with time as kinetic energy is lost. The Chandler wobble decays away over a time span of about 68 years, and so must be continuously excited/driven by some means. Recently Richard Gross, a geophysicist at the Jet Propulsion Laboratory, showed that the principal cause of the Chandler wobble is fluctuating pressure on the bottom of the ocean, caused by temperature and salinity changes and wind-driven changes in the circulation of the oceans. ('The excitation of the Chandler wobble' R.S. Gross RS, Geophysical Reasearch Letters 27, 2329-2332 (2000)).

Explorer 1 Satellite

In 1958, a few months after the Russians launched Sputnik I, the US launched their first satellite Explorer 1, which was a long cylindrical object, with flexible radio antennae protruding from the sides:



Explorer-I satellite. (Drawn by G. Harry Stine.)

To stabilise its orientation it was set spinning about an axis parallel to its length. Unfortunately this is the minor axis of the satellite, and before it had orbited the Earth once, the angular momentum vector had moved to the major axis (perpendicular to the middle of the satellite) and it spent the rest of its mission cart wheeling thought space. Fortunately its instruments and power supply (a battery!) were unaffected by the orientation of the satellite and its mission was a success – discovering the Van Allen radiation belts around the earth.

Lewis Satellite

This satellite was lost in August 1997 shortly after launch – an overview of the causes can be downloaded from http://www.aero.org/capabilities/cords/pdfs/SOPSO-V3-2.pdf.