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Classical Dynamics

J. Ellis


Normal modes: Analysis of motion of many-particle systems in terms of normal modes. Degrees of freedom of a system, matrix notation, orthogonality of the eigenvectors, zero frequency and degenerate modes. Use of symmetries to find normal modes. Modes of molecules and continuous systems (standing waves).


Books

An excellent book, but very expensive so check your College library:

Also good are:
Classical Mechanics, T.W.B. Kibble and F.H. Berkshire (Imperial College Press)

Comprehensive and recommended for 3rd year courses, but above the level of this course:
Classical Mechanics, H. Goldstein, C.P. Poole and J.L.Safko (3rd edition Addison Wesley 2001/3)

For the relevant maths:
Mathematical Methods for Physics and Engineering, Riley R.F., Hobson M.P. and Bence S.J. (CUP 1997)

Acknowledgments
The lecture handouts for this course have drawn on those of the previous lecturers, in particular those of Profs Webber and Mackay and Dr. S. Kenderdine.
1 Frames of reference, Newtonian and Lagrangian Mechanics

1.1 Review of Newtonian Mechanics

1.1.1 Newtonian Mechanics is:

- Non-Relativistic, i.e. \( v << c = 3 \times 10^8 \text{ms}^{-1} \) (so, for example, we do not consider magnetic forces on currents).

- Classical, i.e. \( Et >> h = 6.6 \times 10^{-34} \text{Js} \) (Planck’s constant)

and so assumes that:
- the mass of objects is independent of velocity, time or frame of reference,
- measurements of length and time are independent of the frame of reference and, like measurements of mass, are made by comparison with an arbitrary standard,
- all parameters can be known precisely.

1.1.2 Mechanics

= Statics (absence of motion)
  + Kinematics (description of motion, using vectors for position and velocity)
  + Dynamics (prediction of motion, and involves forces and/or energy)

1.1.3 Statics

- Forces and torques balance.
1.1.4 Kinematics

- motion can be described with respect to different frames of reference, and ‘kinematics’ includes a treatment of the ‘fictitious’ forces that need to be introduced to describe motion in an accelerating frame.

- definition of velocity and acceleration

Velocity: \( v = \lim_{\delta \to 0} \frac{\delta r}{\delta t} \)

Acceleration: \( a = \lim_{\delta \to 0} \frac{\delta v}{\delta t} \)

1.1.5 Dynamics

- Basic principle of dynamics, Newton’s second law [N2]:

\[ ma = F \]

mass of body applied force

1.1.6 Force

- N2 is not a definition of force – force is defined as a push or pull that changes or tends to change a body’s state of rest or relative motion. Force can be measured by various means, the simplest being to compare it (using a balance of some kind) with the force of gravity on a set of standard weights. N2 encapsulates observation that force is a vector.

1.1.7 Mass

- N2 is, however, a definition of inertial mass, which was shown by experiment (summarised in Newton’s Law of Gravitation) to be proportional to gravitational mass.
1.1.8 The use of vectors

- Vectors have both magnitude and direction.

- If a physical quantity has both magnitude and direction and adds, resolves and behaves under coordinate transformations like a vector, it can be usefully represented by a vector.

- Obvious vectors: position, velocity, acceleration, force, momentum

- Not such obvious vectors: quantities that ‘happen’ to behave as vectors and so can be represented as vectors such as angular velocity, angular acceleration, torque, and angular momentum.

1.1.9 Vector basis sets and frames of reference

- Vectors ‘exist’ in space and have properties that are independent of any basis set used to represent them. The same is true of functions of vectors, such as dot and cross products, and derivatives of such as div, grad and curl.

- Vectors may be represented with various basis sets, e.g:

  Cartesian coordinates (basis set of unit vectors independent of position:)

  \[ r = (x, y, z), \text{ where } r = x\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}}, \quad v = (\dot{x}, \dot{y}, \dot{z}) \]

  Cylindrical polar coordinates (basis set varies with position, \( r \)):

  \[ r = (\rho, z), \quad \text{where } r = \rho\mathbf{\hat{\rho}} + z\mathbf{\hat{z}} \]

  \[ v = (\dot{\rho}, \rho\dot{\phi}, \dot{z}) = \dot{\rho}\mathbf{\hat{\rho}} + \rho\dot{\phi}\mathbf{\hat{\phi}} + z\mathbf{\hat{z}} \]

- Coordinate axes (e.g. \( \mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}} \)) and a clock provide a frame of reference, with respect to which every event can be labelled by its space and time coordinates \((r, t)\).

- A frame of reference in which Newton’s laws hold is an inertial frame.
- If a frame S (with coordinates \((x, t)\)) is an inertial frame and frame S’ (with coordinates \((x', t')\)) is in uniform motion relative to S, then S’ is also an inertial frame, because accelerations are the same in both frames. The equations relating the coordinates of the two frames are known as the Galilean transformation:

\[
\begin{align*}
t' &= t \\
x' &= x - R \\
R &= R_0 + ut
\end{align*}
\]

where \(R\) is the position of the origin of the S’ frame in the S frame at time \(t\) and \(u\) is the velocity of S’ with respect to S.

The velocities and accelerations are related by:

\[
\begin{align*}
\dot{x}' &= \dot{x} - \dot{R} = \dot{x} - u \\
\Rightarrow \\
\ddot{x}' &= \ddot{x} = \frac{F}{m}
\end{align*}
\]

\(\Rightarrow\) Newton’s Laws hold in S’ if true in S.

### 1.1.10 Work and Energy

- Work done by a force = force \(\times\) (distance moved parallel to force)

\[
\delta W = F \delta r \cos \theta = F \delta L
\]

- Work done by a force moving along path P

\[
W = \int_{\gamma_1}^{\gamma_2} F \cdot d\gamma = \int_{\gamma_1}^{\gamma_2} F \cdot v \, dt
\]

\(= m \int_{\gamma_1}^{\gamma_2} v \, \frac{dv}{dt} \, dt = \frac{1}{2} \int_{\gamma_1}^{\gamma_2} \frac{dv}{dt} \left( v^2 \right) \, dt \)

\(= \frac{1}{2} m \frac{v_2^2}{v_1^2} - \frac{1}{2} m \frac{v_1^2}{v_1^2} \)

\(=\) increase in KINETIC ENERGY (T)
1.1.11 Potential Energy and Conservative Forces

- If the work done, $W$, by a force in moving along a path from a point $r_1$ to a point $r_2$ is independent of the path taken, then the force is known as a conservative force and it is possible to relate the work done to a function $U(r)$ which takes the form of a potential energy:

$$ W = U(r_1) - U(r_2) $$

(The equation defines $U$ so that if work is done by the force, $U$ decreases)

- equating expressions for the work done in terms of kinetic and potential energies gives:

$$ W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = U(r_1) - U(r_2) $$

$$ \Rightarrow U(r_1) + \frac{1}{2} m v_i^2 = U(r_2) + \frac{1}{2} m v_f^2 \quad \text{Conservation of Energy (} E = T + V \text{)} $$

- 'Conservation of energy' is simply the integral form of N2.

- Other forms of 'energy' (heat, electrostatic etc) are 'fixed' so as to fit in with this simple idea.

- The force may be recovered from the potential energy by differentiating:

1D \[ F(x) = -\frac{dU(x)}{dx} \]

3D \[ F(r) = -\nabla U(r) \]

(\(\nabla\) is the gradient vector operator which is given in Cartesian coordinates by

\[
\nabla U = \left( \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right)
\]

- Central forces are conservative. Suppose:

\[ F(r) = f(r) \hat{r} \]

and since \( \hat{r} \cdot dr = dr \)

we have \[ \int_{r_i}^{r_f} F \cdot dr = \int_{r_i}^{r_f} f(r) dr \]

since the work done by the force in going from $r_1$ to $r_2$ is a scalar integral of $f(r)$, the work done is independent of path and the force is conservative
- How can we know if a force is conservative, without performing integrals over all paths? A force is conservative if:

$$\int F \cdot dr = 0$$

for any closed path. Stokes theorem states that integral of a vector round a closed loop is equal to the integral of the curl of that vector across any surface bounded by that loop. Since for a conservative force $$\int F \cdot dr = 0$$ for all loops then $$\int \nabla \times F \cdot dS = 0$$ for any integration over any area and so for a conservative field:

$$\nabla \times F = 0$$

So, if the curl of a force is everywhere zero, the force is conservative and there exists a scalar function $$\phi(r)$$ such that:

$$F = -\nabla \phi$$

\subsection{1.1.12 Momentum}

- Momentum is defined as the product of mass and velocity:

$$p = mv$$

$$\dot{p} = \frac{dp}{dt} = F$$ \quad \text{(N2 again)}

- If a force acts on a particle from time $$t_i$$ to $$t_f$$, then the time integral of the force (the impulse) is equal to the change in momentum:

$$p_f - p_i = \int_{t_i}^{t_f} F(t) dt$$

- The impulse of a force can be extremely useful in problems such as collisions where the force varies rapidly in an unknown way, but all that is of interest is the ‘net effect’ of the force.
1.1.13 Motion of a collection of particles: the centre of mass

- The centre of mass $\mathbf{R}$ of a collection of particles (positions $\mathbf{r}_i$) is defined as:

$$M \mathbf{R} = \sum_i m_i \mathbf{r}_i$$

where the total mass $M$ is given by:

$$M = \sum_i m_i$$

Using N2, the acceleration of the centre of mass is given by:

$$M \ddot{\mathbf{R}} = \sum_i m_i \ddot{\mathbf{r}}_i = \sum_i \mathbf{F}_i = \sum_i \mathbf{F}_i + \sum_i \sum_j \mathbf{F}_{ij}$$

where $\mathbf{F}_i$ is the total force on the $i^{th}$ particle, $\mathbf{F}_{io}$ is the external force on the $i^{th}$ particle and $\mathbf{F}_{ij}$ is the force on the $i^{th}$ particle from the $j^{th}$. Since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ by Newton’s 3rd law, the term $\sum_i \sum_j \mathbf{F}_{ij}$ sums to zero, and the acceleration of the centre of mass is given by the sum of the external forces $\mathbf{F}_o$:

$$M \ddot{\mathbf{R}} = \sum_i m_i \ddot{\mathbf{r}}_i = \sum_i \mathbf{F}_{io} = \mathbf{F}_o$$

Since $M \ddot{\mathbf{R}} = \mathbf{P}_o$, the total momentum, in the absence of an external force momentum is conserved ($\mathbf{F}_o = \mathbf{P}_o = 0$).
1.2 The Energy Method

- It is often much easier to write an expression for the energy of a mechanical system the to work out the forces acting on the various bodies, and the ‘energy method’ enables single variable problems to be solved, provided energy is conserved.

Example 1.2.1 Simple pendulum (one variable)

Total energy of system:  \( E = T + V \)

Kinetic energy:  \( T = \frac{1}{2} m(l\dot{\theta})^2 = \frac{1}{2} ml^2 \dot{\theta}^2 \)

Potential energy:  \( V = mgy(\theta) = mgl(1 - \cos \theta) \)

\[
\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos \theta) \right) = ml^2 \ddot{\theta} + mgl \sin \theta \dot{\theta} = 0
\]

so, either the system is stationary (\( \dot{\theta} = 0 \)) or:

\[
ml^2 \ddot{\theta} = -mgl \sin \theta
\]

\[
\ddot{\theta} = -\frac{g}{l} \sin \theta
\]

- Note:

(1) the equation of motion is derived from the energy
(2) no forces were considered
(3) one constraint (conservation of energy) was used – so we could solve for only one degree of freedom
Example 1.2.2 1D diatomic molecule

\[ E = \frac{1}{2} m x_1^2 + \frac{1}{2} m x_2^2 + \frac{1}{2} k (x_2 - x_1)^2 \]

Try differentiating w.r.t. time:

\[ \frac{dE}{dt} = m \dot{x}_1 \dot{x}_1 + m \dot{x}_2 \dot{x}_2 + k (x_2 - x_1) (\ddot{x}_2 - \ddot{x}_1) \]

- because there are now two variables – cannot simply divide by the derivative of one of them as in Example 1.2.1

- need an extra constraint to allow one of the variables to be eliminated. Use the fact that since there is no external force – centre of mass moves at constant velocity. We can use coordinates \( X_0 \), the position of the centre of mass and \( X \), the separation between the masses:

\[ X_0 = \frac{1}{2} (x_1 + x_2) \quad X = x_2 - x_1 \quad M = 2m \]

since there is no external force:

\[ \ddot{X}_0 = 0 \]

Which for initial centre of mass position and velocity \( A \) and \( V \) has the solution \( X_0 = A + V t \).

Substituting the new variables into the expression for the energy gives:

\[ E = \frac{1}{2} M \ddot{X}_0^2 + \frac{1}{2} \mu \dddot{X}^2 + \frac{1}{2} kX^2 \quad (\mu \text{ is the reduced mass, } \mu = \left( \frac{m_1 m_2}{m_1 + m_2} \right) = \frac{m}{2}) \]

and taking the derivative with respect to time:

\[ \frac{dE}{dt} = M \ddot{X}_0 \dddot{X}_0 + \mu \dddot{X} \dddot{X} + kX \dddot{X} = 0 \]
since $\ddot{X}_0 = 0$ and if we take the case where $\dot{X} \neq 0$ we can divide by $\dot{X}$ to give the equation of motion:

$$0 = \mu\ddot{X} + kX$$

and we have the solution:

$$X_0 = A + Vt, \quad X = a\cos(\omega t + \epsilon)$$

from which $x_1(t) = X_0 - \frac{X}{2}$ and $x_2(t) = X_0 + \frac{X}{2}$ can be deduced. $(A, V, a$ and $\epsilon$ are constants determined by the state of the molecule at $t = 0$, and $\omega$ is the angular frequency of the simple harmonic motion of $X$, $\omega = \sqrt{\frac{k}{\mu}}$.)
1.3 Lagrangian Mechanics

1.3.1 The need for Lagrangian mechanics

- A straight forward application of N2 to a dynamics problem results in very complex algebra, except in the simplest of cases. Consider the following example:

Example 1.3.1.1 A double pendulum

Two point masses hung from a single point by two threads of each of length \( l \) and tensions \( T_1 \) and \( T_2 \) as indicated. (The \( x \) and \( y \) coordinates are measured w.r.t. the equilibrium position - \( \theta_2 = \theta_1 = 0 \).)

Equations of motion:

\[
\begin{align*}
    m_1 \ddot{x}_1 &= T_2 \sin \theta_2 - T_1 \sin \theta_1 \\
    m_1 \ddot{y}_1 &= T_1 \cos \theta_1 - T_2 \cos \theta_2 - m_1 g \\
    m_2 \ddot{y}_2 &= T_2 \cos \theta_2 - m_2 g
\end{align*}
\]

Constraints needed to eliminate \( T_1, T_2, \theta_1, \theta_2, \) and \( x_1 \) or \( y_1 \) and \( x_2 \) or \( y_2 \):

\[
\begin{align*}
    l^2 &= (l - y_1)^2 + x_1^2 \\
    l^2 &= (l - y_2 + y_1)^2 + (x_2 - x_1)^2 \\
    \sin \theta_1 &= \frac{x_1}{l} \\
    \sin \theta_2 &= \frac{(x_2 - x_1)}{l}
\end{align*}
\]

If we can we find a way of deriving the dynamics from the energy, the problem would be much simpler since one can choose 'easy' variables, and far fewer equations are needed:

\[
\begin{align*}
    \text{K.E.} &= \frac{1}{2} m_1 l^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l^2 \left( \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) \right) \\
    \text{P.E.} &= m_1 gl (1 - \cos \theta_1) + m_2 gl (2 \cos \theta_1 - \cos \theta_2)
\end{align*}
\]

(Note – to find the the velocity of \( m_2 \) the ‘\( \theta_1 \)’ and ‘\( \theta_2 \)’ contributions were added using the ‘cos’ rule.)
1.3.2 Constraints

A constraint is something that links otherwise independent coordinates. As can be seen from Example 1.3.1.1, constraints greatly increase the complexity of dynamics problems, but Lagrangian mechanics can handle them very effectively. There are many types of constraints on a mechanical system, but, for the purposes of Lagrangian mechanics, they can be divided into two types:

**Holonomic constraints** in which the constraint can be expressed as equations connecting the coordinates of the particles (and possibly time). Examples are the two strings in the double pendulum (Example 1.3.1.1), a bead moving along a wire, or particle constrained to stay on a surface.

**Nonholonomic constraints** are those that cannot be expressed by such equations – examples include the walls that constrain particles to lie inside a box, or a particle sliding over a horizontal, solid cylinder, since there comes a point when the particle falls off the cylinder that can only be found once the solution to the motion is known.

1.3.3 The Lagrangian mechanics 'recipe'

For systems that have:

(1) holonomic constraints (and certain extensions to some special nonholonomic cases)

(2) forces can be derived from a scalar potential\(^1\) (which is normally a function of the position of the particles, but in special cases, such as magnetic forces on charged particles, may be a function of velocity as well)

(3) workless constraints (constraint forces perpendicular to allowed motions – the examples of constraints in example 1.3.1.1 are workless)

the following 'recipe' can be used to derive the equations that govern the dynamics of the system:

**Step 1**: Decide on the number of degrees of freedom that the system has \((N_f)\)

**Step 2**: Choose \(N_f\) coordinates \((q_i,\) where the index \(i\) runs from 1 to \(N_f)\) that can be independently varied

**Step 3**: Write down the kinetic energy \((T)\) and potential energy \((V)\) in terms of the coordinates \(q_i\) and their time derivatives \(\dot{q}_i\) and define a quantity, the Lagrangian \((L)\) as:

\[ L = T - V \]
Step 4: For each coordinate $q_i$, an equation of motion is given by:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

(the Euler-Lagrange equations)

Step 5: Solve the $N_f$ equations obtained in step (4) to give the solution to the problem

Comments:

(1) The recipe cannot be used if particles move against frictional forces, since these cannot be described by a scalar potential.

(2) If the problem is in $D$ dimensions, with $N$ particles, and $k$ constraints, there are $ND$ possible independent coordinates, of which $k$ can be eliminated by applying the constraint conditions, giving $N_f = ND - k$. If it makes the problem easier to set up or solve, more coordinates than $N_f$ can be used, but the constraints that would otherwise have been used to eliminate the extra ones must then be included in the Euler-Lagrange equations by the method of Lagrange multipliers (outside the scope of this course).

(3) In defining the state of a system, the $q_i$ and $\dot{q}_i$ coordinates can be specified independently- i.e. you can choose freely where a particle is and how fast it is moving.

(4) A conjugate momentum can be defined that 'belongs' with each coordinate $q_i$:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and the Euler-Lagrange equations become:

$$\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$$

Rate of change of conjugate momentum Generalised force

(The Euler-Lagrange equations are also known simply as the Lagrange equations)
1.3.4 Examples 1: Time independent Lagrangians

Example 1.3.4.1 Simple Harmonic Motion

Consider a mass \( m \) attached by a spring of constant \( k \) to a fixed support.

\[
L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2
\]

The Euler-Lagrange Equation gives:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0
\]

\[
\frac{d}{dt} \left( \frac{1}{2} m \dot{x}^2 \right) - \frac{\partial}{\partial x} \left( -\frac{1}{2} k x^2 \right) = 0
\]

\[
\frac{d}{dt} (m \ddot{x}) + kx = 0
\]

(note how \( \dot{x} \) has been treated as an independent variable, so, e.g. \( \frac{\partial x}{\partial \dot{x}} = 0 \))

\[
m \ddot{x} = -kx
\]

Example 1.3.4.2 Double pendulum

(see figure of Example 1.3.1.1)

\[
L = T - V
= \frac{1}{2} m_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{\theta}_1^2 \cos(\theta_2 - \theta_1) - m_1 g l (1 - \cos \theta_1) - m_2 g l (2 - \cos \theta_1 - \cos \theta_2)
\]

Making the small angle approximation:

\[
L = \frac{1}{2} m_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 + \frac{1}{2} \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} m_1 g l \dot{\theta}_1^2 - \frac{1}{2} m_2 g l \dot{\theta}_2^2
\]

The Euler-Lagrange Equations give:

\[
\theta_1: \quad \frac{d}{dt} \left( m_1 \dot{\theta}_1 + m_2 \dot{\theta}_1 \right) + m_1 g l \dot{\theta}_1 + m_2 g l \dot{\theta}_1 = 0
\]

\[
(m_1 + m_2) g \dot{\theta}_2 + g \dot{\theta}_1 (m_1 + m_2) = 0
\]

1.3.4.1
\[ \theta_2, \quad \frac{d}{dt} \left( m_2 l^2 \ddot{\theta}_2 + m_2 l^2 \dot{\theta}_1 \right) + m_2 g l \dot{\theta}_2 = 0 \]
\[ m_2 l \left( \ddot{\theta}_2 + \ddot{\theta}_1 \right) + m_2 g \theta_2 = 0 \] 

Equations 1.3.4.1 and 1.3.4.2 are the equations of motion of the system, derived with much less effort than a direct application of N2 to the problem.

To solve: for simplicity let \( m_1 = m_2 \) and guess a solution in which \( \theta_2(t) = \alpha \theta_1(t) \). Substituting into Equations 1.3.4.1 and 1.3.4.2 gives:

\[ l(2 + \alpha) \ddot{\theta}_1 + 2 g \theta_1 = 0 \]
\[ l(1 + \alpha) \ddot{\theta}_1 + \alpha g \theta_1 = 0 \]  

Since equations 1.3.4.3 and 1.3.4.4 are now equivalent, the ratio of the coefficients of \( \theta_1 \) and \( \ddot{\theta}_1 \) must be the same in both equations, giving:

\[ \frac{2 + \alpha}{1 + \alpha} = \frac{2}{\alpha} \implies \alpha = \pm \sqrt{2} \]

If we now look for oscillating solutions of the form \( \theta_1 = A \cos(\omega t + \varepsilon) \) we have:

\[ [-\omega^2 l(2 + \alpha) + 2g] A \cos(\omega t + \varepsilon) = 0 \]

and

\[ \omega^2 = \frac{2g}{l(2 + \alpha)} \]

so there are two modes of oscillation,

1\textsuperscript{st} mode: \( \theta_1 \) and \( \theta_2 \) oscillate in phase, \( \theta_2 = \sqrt{2} \theta_1 \) and \( \omega = \frac{2g}{\sqrt{l(2 + \sqrt{2})}} \)

2\textsuperscript{nd} mode: \( \theta_1 \) and \( \theta_2 \) oscillate in antiphase, \( \theta_2 = -\sqrt{2} \theta_1 \) and \( \omega = \frac{2g}{\sqrt{l(2 - \sqrt{2})}} \)

note:

- The Lagrangian method yields the equations of motion, which still need to be solved
- This is a good example of a system showing normal modes – one for each coordinate
Example 1.3.4.3 Double Pulley

\[ T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{y} - \dot{x})^2 + \frac{1}{2} m_3 (\dot{y} + \dot{x})^2 \]

\[ V = -m_1 g x - m_2 g (y - x) + m_3 g (x + y) \]

\[ L = T - V = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{y} - \dot{x})^2 + \frac{1}{2} m_3 (\dot{y} + \dot{x})^2 \]
\[ + m_1 g x + m_2 g (y - x) - m_3 g (x + y) \]

using Euler-Lagrange equations:

for \( x \):
\[ \frac{d}{dt}(m_1 \ddot{x} - m_3 (\dot{y} - \dot{x}) + m_3 (\dot{y} + \dot{x})) = m_1 g - m_2 g - m_3 g \]
\[ \left( m_1 + m_2 + m_3 \right) \ddot{x} + (m_3 - m_2) \ddot{y} = (m_1 - m_2 - m_3) g \]

for \( y \):
\[ \frac{d}{dt}(m_2 (\dot{y} - \dot{x}) + m_3 (\dot{y} + \dot{x})) = m_2 g - m_3 g \]
\[ (m_1 - m_2) \ddot{x} + (m_3 + m_2) \ddot{y} = (m_3 - m_1) g \]

so for \( m_1 = 4 \text{kg}, \ m_2 = 1 \text{kg}, \ m_3 = 3 \text{kg}, \ \text{and} \ g = 10 \text{m/s}^2 \) we have:

\[ 8\ddot{x} + 2\ddot{y} = 0 \]
\[ 2\dddot{x} + 4\dddot{y} = -20 \text{m/s}^2 \]

and \( \dddot{x} = 10/7 \text{m/s}^2 \quad \dddot{y} = -40/7 \text{m/s}^2 \)

Note the tensions of the strings do not appear in the Lagrangian which simplifies the calculation, but by the same token they cannot be derived from the Lagrangian.

1.3.5 Derivation of the Lagrangian mechanics formalism

- The Euler–Lagrange equations are essentially Newton's second law written in generalised coordinates.

- The details of the derivation are rather involved, but effectively one starts with N2 expressed for each Cartesian coordinate of the system, and transforms coordinates from Cartesian to generalised ones, using the fact that the forces can be written as derivatives of a scalar potential, i.e. it is a coordinate transformation – for more details see section 1.6.
1.3.6 The Hamiltonian

- Lagrange’s equations are based on \( q_i \) and \( \dot{q}_i \) as independent coordinates - remember you can choose freely where a particle is and how fast it is moving.

- You may wish to use the conjugate momentum \( p_i \) as variables instead of the velocities \( \dot{q}_i \). If you do, then the same physics is contained within, and expressed by a new function, \( H \), the Hamiltonian, given by:

\[
H = \sum_{i=1}^{N_f} q_i \frac{\partial L}{\partial \dot{q}_i} - L
\]  

1.3.6.1

- Hamilton’s equations of motion are:

\[
\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \quad \quad \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \quad \quad \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t}
\]

(These equations and their use are non examinable).

- In order to describe a system one can either use:

  Lagrange’s equations: \( N_f \) second order equations of motion

  or:

  Hamilton’s: \( 2N_f \) first order equations of motion  (ideal for numerical work)

- A similar variable change – a so called 'canonical transformation' - is found in classical thermodynamics. All the thermodynamical information about a system is contained, for example, in the Gibbs free energy, provided you use \( P \) and \( T \) as base variables, \( i.e. \) within the equation:

\[
dG = VdP - SdT
\]  

1.3.6.2

If you wish to use \( V \) and \( T \) as base variables instead, then in order to have a function that completely describes the system in these new variables, you need to subtract \( PV \) from \( G \) to obtain the Helmholz free energy, \( F = G - PV \), which expresses \( dF \) as an exact derivative with respect to \( V \) and \( T \):

\[
dF = dG - PdV - VdP = -PdV - SdT
\]

If one notes from equation 1.3.6.2 that \( V = \frac{\partial G}{\partial P} \) then we have:
\[ F = G - P \frac{\partial G}{\partial P} \]  

1.3.6.3

which illustrates why \( H \) is derived from \( L \) in the way shown in equation 1.3.6.1.

- If the potential \( V \) is not a function of velocity and if the equations that convert generalised coordinates to Cartesian ones do not contain time explicitly, then it can be shown that the Hamiltonian is equal to the total energy of the system. (For the purists this is a sufficient but not a necessary condition – there are some velocity dependent potentials, such as the one used to describe magnetic forces on charged particles for which the Hamiltonian is still equal to the energy!)

c.f. Example 1.3.4.1

\[
H = \dot{x}m\dot{x} - \frac{1}{2}mx^2 + \frac{1}{2}kx^2 = \frac{1}{2}mx^2 + \frac{1}{2}kx^2 = T + V = E
\]

1.3.7 Examples 2: Time dependent Lagrangians

Example 1.3.7.1 System with SHM driver – driven by a time dependent potential.

Mass on spring driven by moving support. For this system:

\[
T = \frac{1}{2}mx^2 \quad V = \frac{1}{2}k(x - a\cos\omega_0t)^2
\]

and

\[
L = \frac{1}{2}mx^2 - \frac{1}{2}k(x - a\cos\omega_0t)^2
\]

The Euler-Lagrange equation gives:

\[
\frac{d}{dt}(mx\dot{x}) + k(x - a\cos\omega_0t) = 0
\]

\[
mx\ddot{x} + k(x - a\cos\omega_0t) = 0
\]

The Hamilton for the system is time dependent, and in this case equal to the total energy:
\[ H = \dot{x} \frac{\partial L}{\partial \dot{x}} - L = m\dot{x}^2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k(x - a \cos \omega t)^2 \]

**Example 1.3.7.2**  
wire on bead – a system driven by making the actual Cartesian coordinates of the system depend on the generalised one (q the distance of the bead along the wire) in a way that depends on time:

\[ z = q \cos \theta, \quad x = q \sin \theta \cos (\phi), \quad y = q \sin \theta \sin (\phi) \]

\[ T = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} m(q \sin \theta)^2 \dot{\phi}^2 \]

\[ V = mgq \cos \theta \]

\[ L = T - V = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} m(q \sin \theta)^2 \dot{\phi}^2 - mgq \cos \theta \]

The Euler-Lagrange equation gives:

\[ m\ddot{q} - mq \sin^2 \theta \dot{\phi}^2 + mg \cos \theta = 0 \]

\[ \ddot{q} = q \sin^2 \theta \dot{\phi}^2 - g \cos \theta \]

To understand the behaviour of the system, consider how the acceleration along the wire depends on \( q \):

So for \( q > q_0 \), mass starts at rest, and then is flung out with an ever increasing acceleration.

The Lagrangian for this system does not contain time explicitly, and so, as outlined in section 1.3.8 below, the Hamilton for the system is a constant of the motion, given by :

\[ H = \dot{q} \frac{\partial L}{\partial \dot{q}} - L = \dot{q} m\dot{q} - L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m(q \sin \theta)^2 \dot{\phi}^2 + mgq \cos \theta \]
In this case, because the actual Cartesian coordinates of the system depend on the generalised ones in a way that depends on time (a 'rheonomic' system), $H$ is not equal to the total energy which would be given by:

$$E = T + V = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m(q\sin\theta)^2 \dot{\theta}^2 + mgq\cos\theta$$

and given that $H$ is conserved, $E$, with its different functional form, is not conserved.

### 1.3.8 Conservation of energy and the Hamiltonian

- The way $H$ changes with time can be deduced by calculating $\frac{dH}{dt}$:

$$H = \sum_{i=1}^{N_f} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$\frac{dH}{dt} = \sum_{i=1}^{N_f} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) - \frac{dL}{dt}$$ \hspace{1cm} 1.3.6.1

- To work out $\frac{dL}{dt}$ we can start with the expression for $dL$ in terms of changes in the coordinates of $L$, which in general are the $q_i$, the $\dot{q}_i$, and $t$:

$$dL = \sum_{i=1}^{N_f} \frac{\partial L}{\partial q_i} dq_i + \sum_{i=1}^{N_f} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

'Dividing' by $dt$ gives:

$$\frac{dL}{dt} = \sum_{i=1}^{N_f} \frac{\partial L}{\partial q_i} \dot{q}_i + \sum_{i=1}^{N_f} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t}$$

and substituting into equation 1.3.6.1 gives:

$$\frac{dH}{dt} = \sum_{i=1}^{N_f} \dot{q}_i \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] - \frac{\partial L}{\partial t}$$ \hspace{1cm} 1.3.6.2

The Euler-Lagrange equations mean that the square bracket in equation 1.3.6.2 is always zero, and so:
\[
\frac{dH}{dt} = -\frac{\partial L}{\partial t}
\]

So, if \( L \) does not have an explicit time dependence, i.e. if \( t \) does not appear explicitly in \( L \) then \( \frac{\partial L}{\partial t} = 0 \) and \( H \) is a constant of the motion. If in addition, the potential \( V \) is not a function of velocity and if the equations that convert generalised coordinates to Cartesian ones do not contain time explicitly then the Hamiltonian is equal to the energy of the system, and energy is thus conserved. This astonishing result may be expressed thus:

**Energy is conserved for an isolated system because of the time invariance of space.**

(You get the same result if you do the experiment now, or a time \( t \) later, provided you start with an identical initial conditions so \( L \) cannot be a function of the time \( t \) when you start. The isolation of the system ensures that there are no external influences that may themselves vary with time.)

### 1.3.9 Invariance of \( L \) and conservation laws

- The conservation of energy is a specific example of a general principle known as **Noether's Theorem**:

  *If a symmetry exists in the Lagrangian, there is a corresponding constant of the motion*

- What about translational invariance? i.e. what about an isolated system where the Lagrangian of the system does not depend on the location of the system in space – after moving the whole system by an amount \((\delta x, \delta y, \delta z)\) with respect to a Cartesian frame of reference the new Lagrangian \((L')\) is equal to the Lagrangian before the move \((L)\).

\[
L' = L(x + \delta x, y + \delta y, z + \delta z, Q_1, \dot{Q}_1, t) = L + \delta x \frac{\partial L}{\partial x} + \delta y \frac{\partial L}{\partial y} + \delta z \frac{\partial L}{\partial z} + O(\delta x^2)
\]

Since \( L' = L \) for any value of the small displacements \( \delta x, \delta y, \delta z \) the derivatives \( \frac{\partial L}{\partial x} \) etc must be zero. The Euler-Lagrange equations then give:

\[
\frac{d}{dt} p_x = \frac{\partial L}{\partial x} = 0, \ (\text{etc})
\]

and \( p_x, p_y, p_z \) are constants of the motion and linear momentum is conserved. i.e.:

**Linear momentum is conserved because of the translational invariance of space**

- In a similar way the rotational invariance of space leads to the conservation of angular momentum.
1.4 Non-Inertial Frames

1.4.1 Transformations from stationary to rotating frames

- If a point on a rigid body has a position vector \( r \) measured with respect to the axis of rotation, and if the body is rotating with angular velocity \( \omega \), then the velocity of the point with respect to stationary coordinates is given by:

\[
v_0 = \omega \times r
\]

(vectors with the suffix \( _0 \) are with respect to the stationary frame, \( S_0 \), those without a suffix are with respect to the rotating frame, \( S \), and for convenience the equations are written for the instant in time when the coordinate axes of \( S \) and \( S_0 \) coincide, so \( r_0 = r \) and \( \omega_0 = \omega \))

- If the point is also moving on the rigid body with a velocity \( v \) with respect to coordinates that rotate with the body, then this extra velocity must be added into the velocity \( v_0 \):

\[
v_0 = v + \omega \times r
\]

- similarly the time derivative of any vector in \( S_0 \) is related to that of the same vector quantity, but measured with respect to \( S \) by:

\[
\left[ \frac{dA}{dt} \right]_{S_0} = \left[ \frac{dA}{dt} \right]_{S} + \omega \times A
\]

1.4.2 Derivation of fictitious forces

- Newton’s Laws only apply in inertial frames. It is, however, often convenient to solve dynamical problems from the point of view of a rotating (non-inertial) frame such as one fixed to the Earth. If you are making observations in such a frame, and if you insist that you want to use Newton’s laws to describe what you see, then you have to allow for ‘fictitious’ forces that seem to push objects around in mysterious ways.

- In dynamics these ‘forces’ are introduced as a ‘fix’ to make Newton’s laws work in a frame where the laws are actually invalid, because solving the problem in an ‘proper’ inertial frame is harder than worrying about the fictitious (‘fix’) forces in the non-inertial one. They are also introduced because to an observer forces are more apparent than accelerations, and if
something appears to accelerate, we have enough of an intuitive feel for Newton’s laws to ask what force is accelerating them. If you want to hold an object still in a rotating frame, technically what you are doing is accelerating the object towards the axis of rotation. However, what you see is a stationary object and what you feel is the very real centrifugal force (actually a N3 reaction to you pushing and accelerating the object inwards) that tends to fling the object outwards. Furthermore, if you let go of that object, it is a fair question to ask what force produces the apparent acceleration outwards that ensues.

- In order to derive an expression for the acceleration of an object as seen in a rotating frame, we can use the formula given above that rates the time derivative of an vector in stationary and rotating frames. Firstly for velocity:

\[
\frac{dr_0}{dt}_{S_0} = \frac{dr}{dt}_S + \omega \times r, \quad \text{i.e.} \quad v_0 = v + \omega \times r
\]

The subtle bit of this derivation now follows. The true acceleration of an object (i.e. measured in an inertial frame such as S\(_0\) ) is the time derivative of the true velocity, i.e. \(v_0\) if you measure it in S\(_0\) and \(v + \omega \times r\) if you measure it in S. (note the observer in S thinks the velocity is just \(v\), but once he is told that actually it is \(v + \omega \times r\) he can then use this to work out the true acceleration in S coordinates.) So, the expression for the true acceleration, \(a_0\), in terms of vectors measured in S is:

\[
a_0 = \frac{d(v_0)}{dt}_{S_0} = \frac{dv_0}{dt}_S + \omega \times v_0 = \left[ \frac{d(v + \omega \times r)}{dt}_S \right] + \omega \times (v + \omega \times r) \\
= a + \omega \times v + \omega \times v + \omega \times (\omega \times r) \\
= a + 2\omega \times v + \omega \times (\omega \times r)
\]

Now, an observer in the rotating frame measures an acceleration \(a\), and as far as he is concerned, since he wishes to believe that Newton’s laws apply in his non-inertial frame, the total force acting on the object must be given by \(ma\), which will have to be composed as follows:

\[
ma = F - 2m\omega \times v - m\omega \times (\omega \times r)
\]

Total Apparent force  True force = \(ma_0\)  Coriolis force  centrifugal force
- The relation between $a$ and $a_0$ can also be derived by expressing the position vector of a point, $r_0$ explicitly in terms of the coordinates $(x, y, z)$ and rotating basis set, $(\hat{x}, \hat{y}, \hat{z})$ in $S_0$ and taking derivatives w.r.t. time, noting that: $\dot{\hat{x}} = \omega \times \hat{x}$, $\ddot{\hat{x}} = \omega \times (\omega \times \hat{x})$ etc.

$$r_0 = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\dot{r}_0 = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + \ldots + \ldots$$

$$= \dot{x}\hat{x} + x(\omega \times \hat{x}) + \ldots + \ldots$$

$$= \dot{x}\hat{x} + 2\dot{x}(\omega \times \hat{x}) + x[\omega \times (\omega \times \hat{x})] + \ldots + \ldots$$

$$a_0 = a + 2\omega \times v + \omega \times (\omega \times r)$$

1.4.3 Centrifugal force

- Using cylindrical polar coordinates $(\rho, \phi, z)$ (see section 1.1.9) with the axis of rotation along the $z$ axis:

$$r = \rho\hat{\rho} + z\hat{z}, \quad \omega = \omega\hat{\phi}, \quad \rho \text{ is the distance from the axis or rotation}$$

$$\Rightarrow \quad \omega \times r = \omega \rho(\hat{z} \times \hat{\rho}) = \omega \rho \hat{\phi}$$

$$\Rightarrow \quad -\omega \times (\omega \times r) = -\omega^2 \rho(\hat{\phi} \times \hat{\rho}) = \omega^2 \rho \hat{\phi}$$

i.e. Centrifugal force $= m\omega^2 \times$ (distance from axis of rotation)

and is directed away from the axis of rotation

Example: rotation of Earth at equator

$$\Omega^2 R_e = \left( \frac{2\pi}{1 \text{ day}} \right)^2 6400 \text{ km} \approx 0.003 g$$
1.4.4 Coriolis force on Earth’s surface

\[ F_{\text{cor}} = -2m\Omega \times \nu \]

\( \Omega \) = Earth’s angular velocity
\( \nu \) = velocity in frame S', fixed to Earth

To derive the Coriolis force at a latitude \( \lambda \), set up a Coordinate axes in S':

\( \hat{x}' \) pointing East
\( \hat{y}' \) pointing North
\( \hat{z}' \) pointing radially outwards at latitude \( \lambda \).

Expressing the Earth's angular velocity in this coordinate system gives:

\[ \Omega = \Omega \cos \lambda \hat{y}' + \Omega \sin \lambda \hat{z}' \]

for \( \nu \) in the plane of the Earth’s surface at this latitude:

\[ \nu = v_x \hat{x}' + v_y \hat{y}' \]

\[ \Rightarrow \quad F_{\text{cor}} = 2m\Omega \sin \lambda (v_y \hat{x}' - v_x \hat{y}') \quad \text{sideways force} \]

\[ \Rightarrow \quad + 2m\Omega \cos \lambda v_x \hat{z}' \quad \text{vertical force} \]

- The sideways force is always perpendicular to \( \nu \) in surface plane with a magnitude of \( 2m\Omega \nu \sin \lambda \), independent of the direction of \( \nu \) in the surface plane, and acts to the right in the northern hemisphere, and to the left in the southern hemisphere.

- The vertical force is usually small compared to the gravitational attraction of the Earth, and is often neglected.
1.5 Summary of handout 1

N2
• The subject is the outworking of N2
• Integrating N2 w.r.t. space gives conservation of energy
• Integrating N2 w.r.t. time, coupled with N3 gives conservation of momentum

Energy method
• for single coordinate problems taking the time derivative of the total energy yields the equation of motion

Lagrangian mechanics – the basics
• energy type methods are great – avoid dealing with details of forces, - for multi-dimension problems Lagrangian mechanics does the trick.
• Cartesian coordinates are often a nuisance, - Lagrangian mechanics can use the obvious, often non- Cartesian coordinates.
• check the Lagrangain recipe and comments – the main aim of this part of the course is that you get used to some of the basic ideas and can know and use this recipe.

Lagrangan mechanics – the taster of what you can do
• The aim of this part of the course is to give you some idea of the power of Lagrangian mechanics.
• Define the Hamiltonian – note Hamilton’s equations equivalent to the Euler-Lagrange equations, but 1st not 2nd order – easier for computing.
• If the potential is half way normal (i.e. is not a function of velocity) and if things are not stirred up by the relation between Cartesian and generalised coordinates being a function of time – then \( H \) = the energy of the system
\[
\frac{dH}{dt} = -\frac{\partial L}{\partial t}, \text{ so if there is no explicit time dependence in the Lagrangian, } H \text{ is a constant of the motion. If } H \text{ also equals energy, then energy is conserved.}
\]

• The fact that the behaviour of a system is not dependent on the absolute time at which it is started, (i.e. the time invariance of space) means that what we construct and call energy is conserved.
• Noether’s theorem as a generalisation of this.

Fictitious forces
• N2 only works for inertial frames – but is so useful, -let’s try and make it work in non inertial frames (i.e. rotating ones in this context)
• Add in ‘Fictitious’ forces as a fix, then proceed as if N2 were valid.
• Two forces needed:
  • centrifugal, that depends on position: \(-m\omega \times (\omega \times r)\)
  • horizontal component of Coriolis force on Earths surface is \(2m\Omega v \sin \lambda\) perpendicular to motion to the right (left) in the northern (southern) hemisphere.
1.6 Appendix 1: Justification of Lagrange’s equations for non-velocity dependent potentials. (non examinable)

- The purpose of this appendix is to illustrate the fact that the Euler-Lagrange equations are simply the equivalent of N2 in Cartesian coordinates, but written in generalised coordinates.

- For a system of \( N \) interacting bodies there will be \( 3N \) independent coordinates. In an inertial frame with a simple Cartesian coordinate system the equations of motion are given by a direct use of Newton’s second law:

\[
F_i = m_i \ddot{x}_i
\]

For convenience a single index \( i \) is used. One could label the coordinates \( X_1, Y_1, Z_1 \) for the first particle, \( X_2, Y_2, Z_2 \) for the second, with mass \( M_1 \) and \( M_2 \) etc but the maths would be more complicated to write. Here we have assigned:

\[
\begin{align*}
  x_1 &= X_1 \\
  x_2 &= Y_1 \\
  x_3 &= Z_1 \\
  x_4 &= X_2 \\
  x_5 &= Y_2 \\
 &\text{etc}
\end{align*}
\]

and for convenience the mass associated with a particular coordinate is given the same subscript as the index coordinate:

\[
\begin{align*}
  m_1 &= M_1 \\
  m_2 &= M_1 \\
  m_3 &= M_1 \\
  m_4 &= M_2 \\
  m_5 &= M_2 \\
 &\text{etc}
\end{align*}
\]

However, in most systems, it is often algebraically complicated to set up a dynamics problem in terms of the basic Cartesian coordinates, in the way given in equation 1.6.1, and there are often much more natural variables to choose. What we want to do is to find is how N2 (equation 1.6.1) is written in non Cartesian coordinates. The result is going to be the Euler-Lagrange equations.

\[
\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}
\]

Rate of change of conjugate momentum

Generalised force
In Cartesian coordinates, and for a conservative potential $V$, $N_2$ takes the form

$$m_i \dddot{x}_i = -\frac{\partial V}{\partial x_i}$$

and the kinetic energy $T = \sum \frac{1}{2} m_i \dot{x}_i^2$

The generalised coordinates of the system, $q_i$, can be written as functions of the Cartesian coordinates.

$$q_i = q_i(x_1, x_2, x_3, \ldots x_j, \ldots x_{n_j})$$

The Euler-Lagrange equations take the form:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

For a conservative potential, $V$ is independent of the velocities of the particles – both in Cartesian and generalised coordinates, i.e.

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_i}$$

We are therefore trying to prove that

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) = \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i}$$

Consider each term:

$$\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \left( \sum j \frac{1}{2} m_j \dot{x}_j^2 \right) = \sum j m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i}$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left( \sum j m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i} \right) = \sum j m_j \dddot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i} + \sum j m_j \dot{x}_j \frac{d}{dt} \left( \frac{\partial \dot{x}_j}{\partial \dot{q}_i} \right)$$

1.6.2
\[ \frac{\partial T}{\partial q_i} = \sum_j m_j \dot{x}_j \left( \frac{\partial x_j}{\partial q_i} \right) \]

\[ \frac{\partial V}{\partial q_i} = \sum_j \frac{\partial V}{\partial x_j} \left( \frac{\partial x_j}{\partial q_i} \right) = \sum_j -m \ddot{x}_j \left( \frac{\partial x_j}{\partial q_i} \right) \]

\[ \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} = \sum_j m_j \dot{x}_j \left( \frac{\partial x_j}{\partial q_i} \right) + \sum_j m_j \ddot{x}_j \left( \frac{\partial x_j}{\partial q_i} \right) \quad 1.6.3 \]

so if the right hand sides of equations 1.6.2 and 1.6.3 are equal, i.e. if

\[ \frac{d}{dt} \left( \frac{\partial x_j}{\partial q_i} \right) = \frac{\partial x_j}{\partial q_i} \quad \text{and} \quad \frac{\partial x_j}{\partial q_i} = \frac{\partial x_i}{\partial q_i} \]

then so are the left and sides and we have proved that the Euler-Lagrange equations are equivalent to N2 for a many coordinate system in non-Cartesian coordinates.

Consider:

\[ dx_j = \sum_i \frac{\partial x_j}{\partial q_i} dq_i \quad \Rightarrow \quad \frac{dx_j}{dt} = \sum_i \frac{\partial x_j}{\partial q_i} dq_i \quad \Rightarrow \quad \dot{x}_j = \sum_i \frac{\partial x_j}{\partial q_i} \dot{q}_i \]

and since \( x_j \) is not a function of \( \dot{q}_i \), neither is \( \frac{\partial x_j}{\partial q_i} \) and so:

\[ \frac{\partial \dot{x}_j}{\partial q_i} = \frac{\partial x_j}{\partial q_i} \]

as required. Furthermore consider

\[ \frac{d}{dt} \left( \frac{\partial x_j}{\partial q_i} \right) = \frac{d}{dt} \left( \frac{\partial x_j}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left( \frac{dx_j}{dt} \right) = \frac{\partial x_j}{\partial q_i} \quad \text{again as required.} \]