Lent Term 2005

1 The number of gas atoms, with speed c to c+dc, striking unit area, in unit time is $\frac{1}{4} n\, c\, f(c)\mathrm{d}c$, where n is the number density of atoms and the probability distribution, $f(c)\mathrm{d}c$, for c is

$$f(c)\,\mathrm{d}\,c = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mc^2}{2kT}\right) 4\pi c^2\,\mathrm{d}\,c\,,$$

determine the number of atoms, per second, passing through a small hole in the gas container, by integrating over all c. Obtain the total kinetic energy of escaping atoms and hence show the mean kinetic energy of atoms leaving the container is $2kT$. Comment on why the result exceeds the mean kinetic energy of the gas atoms. $\left[\int_o^\infty x^n e^{-x} dx = n!\right]$

2 A box has a partition, which divides its volume in the ration 3:1. Initially the smaller region contains 100 atoms. Determine the most likely distribution of atoms between the two regions if a small hole is made in the partition so that the two regions can come into equilibrium. Compare the probability of the equilibrium arrangement with the probability for a recurrence of the original arrangement

3 (a) By considering some small numbers N and m, verify that the number of ways (degeneracy) of sharing m quanta, each of energy ε, amongst N oscillators is

$$g(N,m) = (N+m-1)! \,/\, \{(N-1)!\; m!\,\}$$

(b) Enumerate the degeneracy of a system of 4 oscillators and a system of 2 oscillators, for up to 5 quanta each.

(c) Consider 3 systems, A, B and C, in thermal contact. The systems A and B each consist of 4 oscillators and system C consists of 2 oscillators. What are the most likely ways in which 5 quanta will be shared when the three systems are in equilibrium? Comment on your answer.

(d) Two systems containing large numbers of weakly interacting oscillators N_1 and N_2 are in thermal contact *i.e.* able to exchange energy with one another. If the total energy available to be shared amongst the $N_1 + N_2$ oscillators is fixed at $m\varepsilon$, obtain an expression for $g(m_1)$, the number of ways in which system 1 has energy $m_1\varepsilon$ and system 2 has energy $m_2\varepsilon$ where $m_2 = m - m_1$.

By evaluating the first and second derivatives of g w.r.t. m_1, show that the most probable value of m_1 satisfies the relation $m_1 / N_1 = m_2 / N_2$ and that the sharing of energy between the two systems will be very sharply peaked at this value.

4 In thermal equilibrium at temperature T, a crystal with N atoms can have a certain number n of lattice sites vacant. Obtain an expression for the number of ways, g, in which this situation can arise by considering the different arrangements of the n vacant sites chosen from N possibilities.

Assuming that $S = k\ln g$ and that the internal energy $U = n\varepsilon$, find an expression for the vacancy concentration n/N by minimising the free energy $F = U - TS$. Estimate the vacancy concentration in copper at 1300K if $\varepsilon = 1$eV.

5 Initially, the probability that a system is in state X is x_0 and the probability that it is in state Y is $y_0 = 1-x_0$. The probability of a transition from X to Y or from Y to X is α per unit time. Thus, $\mathrm{d}x/\mathrm{d}t = \alpha(-x+y)$ and $\mathrm{d}y/\mathrm{d}t = \alpha(x-y)$.

Show that, after a time greater than a few times $1/\alpha$, x and y approach equality.

What has this to do with statistical mechanics?

6 A box contains a large number, N, of non-interacting 2-state systems, each of which can have energy 0 or ε. The box is in thermodynamic equilibrium with a heat reservoir at temperature T. Draw labelled diagrams showing (i) the energy and (ii) the heat capacity of the box as a function of T.

7 Molecules adsorbed onto the surface of a solid can behave like a two-dimensional gas, so long as they are not too crowded. Assuming that they behave like an ideal gas, show that the mean kinetic energy per molecule is kT and find the distribution of molecular speeds (analogous to the Maxwellian distribution for a 3D gas).

8 When $kT >> m_0c^2$ (where m_0 is the rest mass of the particles) the particles in a plasma move with speeds close to c , and their energies ε are related to their momenta $p = \hbar k$ by $\varepsilon = \{ m_0^2c^4+p^2c^2\}^{1/2} \approx pc$. Show that at such high temperatures the mean particle energy is $3kT$ (as compared with $(3/2)\, kT$ for non-relativistic gases). At what temperatures would this approximation become relevant for (a) electrons (b) protons?

9 Radiation at frequency ω stimulates atomic (or molecular) transitions up from state 1 (energy E_1) to state 2 (energy $E_2 = E_1 + \hbar\omega$) and also transitions from state 2 to state 1. It follows from Fermi's Golden Rule that the rate of stimulated transitions is proportional to the energy density $u(\omega)$ of the radiation (at the transition frequency) and is the same in both directions:

$$dn_2/dt = Bun_1 - Bun_2$$

where B is a constant for that transition.

By considering atoms in thermal equilibrium in an enclosure at temperature T , show that, in addition to the stimulated transitions, there must be spontaneous transitions downward at a rate A per excited atom per second where $A = \hbar\omega^3B/\pi^2c^3$. (They are called the Einstein A and B coefficients.)

10 Calculate the entropy of the lattice vibrations of a solid as described by the Einstein theory. Obtain limiting expressions for the entropy valid at low and at high temperatures.

11 Single-electron paramagnetism was considered in the lectures; there, the spin and magnetic moment can be either Up or Down with respect to the magnetic field. Ions with larger total angular momentum can take up more different orientations. The extreme is the classical limit, in which the magnetic moment can point in any direction. Show that in this limit the magnetic susceptibility is $\mu_o m^2 /3kT$ per ion, where m is the magnitude of the magnetic dipole-moment

12 A magnetic material, in one-dimension, can be modelled as a linear array of N+1 spins.

$$\downarrow\downarrow\downarrow \,\vdots\, \uparrow\uparrow\uparrow\uparrow \,\vdots\, \downarrow\downarrow \,\vdots\, \uparrow\uparrow\uparrow\uparrow\uparrow \,\vdots\, \downarrow\downarrow\downarrow$$

Domain walls separate regions with spin up and spin down. Calculate the number of ways n domain walls can be arranged. Deduce the entropy, $S(n)$. If each domain wall has energy, ε, the energy of the system is given by $E = n\varepsilon$. Show that the energy is related to the temperature by

$$E = N\varepsilon/\left[\exp(\varepsilon/kT)+1\right]$$

Sketch the energy of the system and its heat capacity, as functions of T. Comment on the behaviour and give sketches of the state of the system at high and low temperatures. (Hint: Temperature and entropy are related through $(1/T) = \partial S/\partial E$.)
(Based on IB Adv. Physics, 1993, Paper 2, C11)

13 Two electrons share a potential well, in which each can have energy 0 or ε. By Pauli's exclusion principle they cannot both be in the same quantum state, so if, for example, both are in the same space state of energy ε then one must have its spin up, the other its spin down. Find an expression for the probability that the total spin in the z -direction is not zero, and sketch its dependence on temperature.

Numerical answers:

2. 75:25, ~10^{59} times more likely
3. (c) A/B/C: 2/2/1; 3/2/0; 2/3/0
4. 1.3×10^{-4}
5. 8.(a) 2×10^{9} K; (b) 4×10^{12} K